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# Path–Path Ramsey-Type Numbers for the Complete Bipartite Graph

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For a fixed pair of integers $r, s \geqslant 2$, all positive integers $m$ and $n$ are determined which have the property that if the edges of $K_{m,n}$ (a complete bipartite graph with parts $n$ and $m$) are colored with two colors, then there will always exist a path with $r$ vertices in the first color or a path with $s$ vertices in the second color.

Recently, Ramsey numbers have been investigated for various pairs of graphs $(G_1, G_2)$ (see [1–9, 11–13]). Very little has been done for more general Ramsey numbers, those involving multicolorings of the complete graph. To solve what should be the simplest of the multicolorings, namely, finding the Ramsey number for paths under three colorings, a special problem arises. This problem involves finding the "smallest" complete bipartite graph which when two colored contains a path of length $i$ in the first color or of length $j$ in the second color. The solution of this problem is the content of this paper. Of course, this suggests a whole area of Ramsey-type problems involved in two coloring special graphs other than the complete ones.

The "smallest" two colored complete bipartite graph for a pair of paths will be determined by what is called the Ramsey bipartite number pair in the sequel. Notation will be in terms of a graph and its complement (relative to a complete bipartite graph) rather than in terms of two colorings.

All graphs will as usual be undirected, finite, and have no loops or multiple edges. $K_r$ will denote the complete graph on $r$ vertices and $K_{r,s}$ the complete bipartite graph with parts containing $r$ and $s$ vertices, respectively. If $G$ is a graph, $V$ will denote its vertex set and $E$ its edge set. Notation not specifically stated will follow that in [10].

A graph $G$ is an $(n, m)$ graph if it is a subgraph of $K_{n,m}$ and is of order $n + m$. Thus if $G$ is an $(n, m)$ graph, there exist disjoint sets $N$, $M$ of $V$

(called the parts of $G$) with $|N| = n$ and $|M| = m$. If we wish to discuss the parts of $G$ we will denote $G$ by $G_{N,M}$. In this paper all graphs $G$, unless otherwise indicated, will be $(n, m)$ graphs, and $\bar{G}$ will denote the complement of $G$ relative to $K_{n,m}$. In keeping with this definition, if $G = K_{n,m}$, then $\bar{G}$ will denote the graph with the same vertex set as $G$ having no edges.

For $A \subseteq V$ and $u \in V$, let $A_u = \{a \in A \mid (a, u) \in E\}$ and $d_A(u) = |A_u|$. If $A = V$, $A$ may be deleted and $d_A(u)$ may be written as $d(u)$ or $d_G(u)$. A bar will be placed over a symbol when in the graph $\bar{G}$ instead of $G$, for example, $\bar{E}$ is the edge set of $\bar{G}$. A path $P_l$ with $l$ vertices $\{x_1, x_2, \ldots, x_l\}$ will be denoted by $(x_1, x_2, \ldots, x_l)$ while a cycle $C_l$ with the same set of vertices will be written $(x_1, x_2, \ldots, x_l, x_1)$, the indices taken modulo $l$.

Let $G_1$ and $G_2$ be usual graphs (subgraphs of some $K_r$, not necessarily of some $K_{r,s}$, as described above). Let the ordered pair of positive integers $(n, m)$, $n \geqslant m$, be such that each $(r, s)$ graph $G$, $r \geqslant s$, contains a subgraph $G_1$ or its complement $\bar{G}$ contains a $G_2$ if and only if $r \geqslant n$ and $s \geqslant m$. Such a pair $(n, m)$, if it exists, will be called the *Ramsey bipartite number pair* for the ordered pair of graphs $(G_1, G_2)$ and will be denoted by $B(G_1, G_2) = (n, m)$. Surely, when it exists, $B(G_1, G_2) = B(G_2, G_1)$. In this paper we determine $B(P_i, P_j)$ for all $i, j \in Z^+$.

An $(n, m)$ graph $G = G_{N,M}$ will be said to *separate* if there exist disjoint nonempty sets $A$, $B$, $C$, $D$ and graphs $G'$, $G''$ such that $G_{N,M} = G_{A\cup C, B\cup D} = G'_{A,B} \cup G''_{C,D}$. The graphs $G'_{A,B}$ and $G''_{C,D}$ will be called components (not necessarily connected) of the separated graph.

If in addition $G'_{A,B} \cong K_{i,j}$, and $G''_{C,D} \cong K_{l,k}$, $G = G_{N,M}$ will be said to *split into the pair* $(K_{i,j}, K_{l,k})$. When $G$ splits, we allow some of $i, j, k, l$ to be zero, e.g., if $k = 0$ then $G$ is an $(i + l, j + k)$ graph with $K_{i,j}$ as a subgraph, and if $k = l = 0$ then $G$ is a complete bipartite graph. If the pair into which $G$ splits is not important, we will simply say that the graph $G$ splits. Also observe that $G$ splitting into the pair $(K_{i,j}, K_{l,k})$ is identical to $\bar{G}$ splitting into the pair $(K_{i,k}, K_{l,j})$, so that $G$ splits if and only if $\bar{G}$ splits.

The main result of this paper is

THEOREM. *For* $n, m \in Z^+$,

(i) $B(P_{2n}, P_{2m}) = (n + m - 1, n + m - 1)$,

(ii) $B(P_{2n+1}, P_{2m}) = (n + m, n + m - 1)$ *for* $n \geqslant m - 1$,

(iii) $B(P_{2n+1}, P_{2m}) = (n + m - 1, n + m - 1)$ *for* $n < m - 1$,

(iv) $B(P_{2n+1}, P_{2m+1}) = (n + m, n + m - 1)$ *for* $n \neq m$,

(v) $B(P_{2n+1}, P_{2n+1}) = (2n + 1, 2n - 1)$.

The proof of this result will be broken into several lemmas and theorems.

LEMMA 1. *Let $G = G_{N,M}$, $y_1 \in N$, $P = (x_1, x_2, \ldots, x_j)$, $j \geqslant 2$, a path in $G \backslash \{y_1\}$, and $Q = (y_1, y_2, \ldots, y_k)$, $2 \leqslant k \leqslant 4$, a path in $G \backslash P$.*

*If $j$ is even and $x_1 \in M$, then at least one of the following occurs.*

(1) *$G$ separates, with $\{y_1, y_2\}$ and $P$ in different components;*

(2) *$G$ contains a $P_{j+k-1}$;*

(3) $d_P(x_1) < \bar{d}_P(y_1)$.

*If $j$ is odd, then*

(1) *$x_1 \in M$ implies $G$ contains a $P_{j+k}$ or $d_P(x_1) < \bar{d}_P(y_1)$;*

(2) *$x_1 \in N$ implies $G$ contains a $P_{j+k-1}$ or $d_P(x_1) \leqslant \bar{d}_P(y_1)$.*

*Proof.* Let $j$ be even, $x_1 \in M$, with neither $G$ separating, as indicated in (1), nor $G$ containing a $P_{j+k-1}$. We first show that $(y_1, x_{j-1}), (x_1, x_j) \in \bar{E}$. Now $(y_1, x_{j-1}) \in \bar{E}$, otherwise $(y_k, y_{k-1}, \ldots, y_1, x_{j-1}, x_{j-2}, \ldots, x_1)$ is a $P_{j+k-1}$ in $G$. Also if $(x_1, x_j) \in E$, then, since $G$ does not separate with $\{y_1, y_2\}$ and $P$ in different components, there exists a path from a point of $Q$ to a point of the cycle $(x_1, x_2, \ldots, x_j, x_1)$. From this path, points of $Q$, and points of the cycle, we get a $P_q$, $q \geqslant j + k - 1$ (recall that $2 \leqslant k \leqslant 4$), a contradiction. Hence $(y_1, x_{j-1}), (x_1, x_j) \in \bar{E}$. Also if $(x_1, x_l) \in E$, then $(y_1, x_{l-1}) \in \bar{E}$, $2 \leqslant l \leqslant j - 1$; otherwise $(y_k, y_{k-1}, \ldots, y_1, x_{l-1}, x_{l-2}, \ldots, x_1, x_l, x_{l+1}, \ldots, x_j)$ is a $P_{j+k}$ in $G$. Thus $d_P(x_1) \leqslant \bar{d}_P(y_1)$. But $(y_1, x_{j-1})$, $(x_1, x_j) \in \bar{E}$, so that this inequality is strict, i.e., $d_P(x_1) < \bar{d}_P(y_1)$.

Next let $j$ be odd with $x_1 \in M$, and assume $G$ contains no $P_{j+k}$. Thus $(y_1, x_1), (y_1, x_j) \in \bar{E}$, otherwise $Q \cup P$ with edge $(y_1, x_1)$ or $(y_1, x_j)$ form a $P_{j+k}$ in $G$. Also if $(x_1, x_l) \in E$, then $(y_1, x_{l-1}) \in \bar{E}$, $2 \leqslant l \leqslant j - 1$. Thus just as before, $d_P(x_1) \leqslant \bar{d}_P(y_1)$. The strict inequality follows since $(y_1, x_j) \in \bar{E}$.

Finally, let $j$ be odd with $x_1 \in N$ and assume $G$ contains no $P_{j+k-1}$. If $(x_1, x_l) \in E$, then $(y_1, x_{l-2}) \in \bar{E}$, $4 \leqslant l \leqslant j - 1$, otherwise $(y_k, y_{k-1}, \ldots, y_1, x_{l-2}, x_{l-3}, \ldots, x_1, x_l, x_{l+1}, \ldots, x_j)$ is a $P_{j+k-1}$ in $G$. Also $(y_1, x_{j-1}) \in \bar{E}$, otherwise $(y_k, y_{k-1}, \ldots, y_1, x_{j-1}, x_{j-2}, \ldots, x_1)$ is a $P_{j+k-1}$ in $G$. Since $(y_1, x_{j-1}) \in \bar{E}$, $(x_1, x_2) \in E$ and $(x_1, x_l) \in E$ implies $(y_1, x_{l-2}) \in \bar{E}$, $4 \leqslant l \leqslant j - 1$, we have $d_P(x_1) \leqslant \bar{d}_P(y_1)$.

LEMMA 2. *Let $y_1, y_2 \in V$, $(y_1, y_2) \in E$, and $P = (x_1, x_2, \ldots, x_j)$, a maximal path in $G \backslash \{y_1, y_2\}$. For $i = 1, 2$, let $\delta_i = 1$ if there exists $z_i \in G \backslash (\{y_1, y_2\} \cup P)$ such that $(y_i, z_i) \in E$, and $\delta_i = 0$ otherwise. Let $\delta = 1 + \min\{1, \delta_1 + \delta_2\}$ and $\mu = 1 + [(\delta_1 + \delta_2)/2]$.*

*If $j$ is even then at least one of the following occurs.*

(1) *$G$ separates with $\{y_1, y_2\}$ and $P$ in different components,*

(2) *G contains a* $P_{j+\mu+1}$,

(3) $d(x_1) + d(x_j) \leqslant \bar{d}(y_1) + \bar{d}(y_2)$ *and* $(x_1, x_j) \in \bar{E}$.

*Furthermore, if* (1) *does not hold and G does not contain a* $P_{j+\delta}$, *then the inequality of* (3) *becomes strict.*

*If j is odd, then at least one of the following occurs.*

(1) *G separates with* $\{y_1, y_2\}$ *and P in different components,*

(2) *G contains a* $P_{j+\mu}$,

(3) $d(x_1) + d(x_{j-1}) < \bar{d}(y_1) + \bar{d}(y_2)$ *and* $(x_1, x_{j-1}) \in \bar{E}$.

*Proof.* Let $j$ be even, $G = G_{N,M}$, $y_1, x_j \in N$, $y_2, x_1 \in M$, and assume neither (1) nor (2) holds. If $(x_1, x_j) \in E$, then the vertices of $P$ form a cycle $C$. Since (1) does not occur, there exists a path between the component containing $\{y_1, y_2\}$ and $C$. Hence $G$ contains a $P_{j+\mu+1}$, a contradiction. Thus $(x_1, x_j) \in \bar{E}$. Also if $(x_1, x_l) \in E$, $2 \leqslant l \leqslant j$, we have $(y_1, x_{l-1}) \in \bar{E}$. Otherwise $(y_2, y_1, x_{l-1}, x_{l-2}, \ldots, x_1, x_l, x_{l+1}, \ldots, x_j)$ [respectively, $(z_2, y_2, y_1, x_{l-1}, x_{l-2}, \ldots, x_1, x_l, x_{l+1}, \ldots, x_j)$] is a $P_{j+\mu+1}$ in $G$ for $\mu = 1$ [respectively, $\mu = 2$]. Hence $d_P(x_1) \leqslant \bar{d}(y_1)$. But $P$ a maximal path in $G \backslash \{y_1, y_2\}$ and $G$ containing no $P_{j+\mu+1}$ imply $d_P(x_1) = d(x_1)$. Hence $d(x_1) \leqslant \bar{d}(y_1)$. Similarly, $d(x_j) \leqslant \bar{d}(y_2)$, so that (3) holds.

Next, for $j$ even, assume (1) does not hold and that $G$ does not contain a $P_{j+\delta}$. We show that the inequality of (3) is strict. Applying Lemma 1 with $k = \delta + 1$, we have $d_P(x_1) = d(x_1) < \bar{d}_P(y_1)$. Also if $(x_j, x_l) \in E$, $1 \leqslant l \leqslant j - 1$, then $(y_2, x_{l+1}) \in \bar{E}$. Otherwise $(y_1, y_2, x_{l+1}, x_{l+2}, \ldots, x_j, x_{l-1}, \ldots, x_1)$ is at least a $P_{j+\delta}$ in $G$. Then $d_P(x_j) = d(x_j) \leqslant \bar{d}(y_2)$ and $d(x_1) + d(x_j) < \bar{d}(y_1) + \bar{d}(y_2)$.

Finally, let $j$ be odd, $G = G_{N,M}$, $y_1 \in M$, and without loss of generality, $x_1, x_j \in N$. Assume that (1) does not hold and $G$ contains no $P_{j+\mu}$. It then follows that $(x_1, x_{j-1}) \in \bar{E}$; otherwise from the sets $\{x_1, x_2, \ldots, x_{j-1}\}$ and $\{y_1, y_2\}$ (or $\{y_1, y_2, z_1, z_2\}$ when $\mu = 2$), and a path joining these sets, we get a $P_{j+\mu}$ in $G$. Also, since $G$ contains no $P_{j+\mu}$, $(y_2, y_2)$, $(y_2, x_{j-l}) \in \bar{E}$. If $(x_{j-1}, x_l) \in E$, $3 \leqslant l \leqslant j - 2$, then $(y_2, y_{l+1}) \in \bar{E}$; otherwise we get a $P_{j+\mu}$ in $G$. Using this with the fact that $(x_1, x_{j-1})$, $(y_2, x_2) \in \bar{E}$, we have $d_P(x_{j-1}) \leqslant \bar{d}(y_2)$. Finally, since $G$ contains no $P_{j+\mu}$ and $(y_2, x_{j-1}) \in \bar{E}$, $d_{G \backslash P}(x_{j-1}) \leqslant \bar{d}_{G \backslash P}(y_1)$. Also by Lemma 1, $d(x_1) = d_P(x_1) < \bar{d}_P(y_1)$. Hence

$$\begin{aligned} d(x_1) + d(x_{j-1}) &= d(x_1) + d_P(x_{j-1}) + d_{G \backslash P}(x_{j-1}) \\ &< \bar{d}_P(y_1) + \bar{d}_{G \backslash P}(y_1) + \bar{d}(y_2) = \bar{d}(y_1) + \bar{d}(y_2). \end{aligned}$$

LEMMA 3. *Let G be an* $(n + m - 1, n + m - 2)$ *graph,* $n, m \geqslant 2$. *If* $(x, y) \in E$ *and* $G \backslash \{x, y\}$ *splits, then at least one of the following occurs.*

(1) *G contains a* $P_{2n}$,

(2) $\bar{G}$ *contains a* $P_{2m}$,

(3) *G splits.*

*Proof.* Let $G = G_{N,M}$ and $x \in N$, $y \in M$. Furthermore, assume $G\backslash\{x, y\}$ splits into the pair $(G_{A,B}, G_{C,D})$. Assume neither (1) nor (2) holds. Since (1) does not hold, $\min\{|A|, |B|\}$, $\min\{|C|, |D|\} \leqslant n - 1$, and since (2) does not hold, $\min\{|A|, |D|\}$, $\min\{|B|, |C|\} \leqslant m - 1$.

Consider the case $n > m + 1$. We cannot have both $|A| \leqslant m - 1$ and $|D| \leqslant m - 1$; thus there is no loss of generality in assuming $|D| \leqslant m - 1$. This implies $|B| \geqslant n - 2$, $|C| \leqslant m - 1$, and $|A| \geqslant n - 1$.

Assume $|B| = n - 2$. Then $|D| = m - 1$. Under these conditions, $(y, a) \in E$ for all $a \in A$; otherwise $\bar{G}$ has a $P_{2m}$. This implies $(x, d) \in \bar{E}$ for all $d \in D$; otherwise $G$ contains a $P_{2n}$. Therefore, since $\bar{G}$ contains no $P_{2m}$ and $G$ contains no $P_{2n}$, $(x, b) \in E$ for all $b \in B$ and $(y, c) \in \bar{E}$ for all $c \in C$. This implies that $G$ splits.

Assume $|B| \geqslant n - 1$. Since $G$ does not contain a $P_{2n}$, $(x, b), (y, a) \in \bar{E}$ for all $a \in A$ and $b \in B$. If $(x, d) \in \bar{E}$ or $(y, c) \in \bar{E}$ for some $c \in C$ or some $d \in D$, then there exists a path in $\bar{G}$ containing $2|D| + 2|C| + 4$ vertices which involve $C \cup D \cup \{x\} \cup \{y\}$ and some vertices of $A$ and $B$. By assumption $2|D| + 2|C| + 4 < 2m$, and hence $|D| + |C| \leqslant m - 3$. Therefore $|A| + |B| \geqslant 2n + m - 2$, a contradiction to $\min\{|A|, |B|\} \leqslant n - 1$. This implies $(x, d) \in E$ and $(y, c) \in E$ for all $d \in D$ and $c \in C$. Thus $G$ splits.

The case analysis for the remaining possibilities are similar and thus are not included here.

LEMMA 4. *Let G be an* $(n + m - 1, n + m - 2)$ *graph,* $n, m \geqslant 2$, *which splits. If G has no* $P_{2n}$ *and* $\bar{G}$ *no* $P_{2m}$, *then* $G = (K_{t,n-1}, K_{v,m-1})$, *where* $0 \leqslant v \leqslant \min\{n - 1, m - 1\}$.

*Proof.* For convenience, we assume $n \geqslant m$, since the argument is the same for $m \geqslant n$.

Suppose $G = (K_{q,r}, K_{p,s})$, where $r > n - 1 \geqslant m - 1$. Then $q \leqslant n - 1$, otherwise $G$ contains a $P_{2n}$; and $p \leqslant m - 1$, otherwise $\bar{G}$ contains a $P_{2m}$. Hence $q + p \leqslant n + m - 2$, a contradiction. If $n - 1 > r, s > m - 1$, then, since one of $q$ or $p$ is strictly larger than $m - 1$, $\bar{G}$ contains a $P_{2m}$. Thus we assume $r = n - 1$ and $s = m - 1$. Now if $m < n$, then $0 \leqslant p \leqslant m - 1$; otherwise $\bar{G}$ contains a $P_{2m}$. If $n = m$, the bounds on $v$ give all possibilities.

LEMMA 5. *Let* $G = G_{N,M}$ *separate with components* $G'_{A,B}$ *and* $G''_{C,D}$.

(1) *Let* $|N| = n + m - 1$, $|M| = n + m - 2$ $(n, m \geqslant 1, n + m \geqslant 3)$, *and assume* $B(P_{2i}, P_{2j}) = (i + j - 1, i + j - 1)$ *for* $i, j \geqslant 1$ *and* $i + j < n + m$. *Then at least one of the following occurs.* (i) $G$ *contains a* $P_{2n}$, (ii) $\bar{G}$ *contains a* $P_{2m}$, (iii) $G$ *splits.*

(2) *Let* $|N| = n + m$, $|M| = n + m - 1$, $n \neq m$, *and assume that each* $(i + j, i + j - 1)$ *graph contains a* $P_{2i+1}$ *or its complement contains a* $P_{2j}$. *Then* $G$ *contains a* $P_{2n+1}$ *or* $\bar{G}$ *contains a* $P_{2m+1}$.

(3) *Let* $|N| = 2n + 1$, $|M| = 2n - 1$, *and assume* $B(P_{2i+1}, P_{2j}) = (i + j, i + j - 1)$ *for* $0 \leqslant j - 1 \leqslant i$, *and* $i \geqslant 1$. *Then* $G$ *or* $\bar{G}$ *contains a* $P_{2n+1}$.

(4) *Let* $|N| = n + m - 1 = |M|$ $(m - 1 > n \geqslant 1)$ *and assume* $B(P_{2i}, P_{2j}) = (i + j - 1, i + j - 1)$ *for* $i, j \geqslant 1$. *Then* $G$ *contains a* $P_{2n+1}$ *or* $\bar{G}$ *a* $P_{2m}$.

*Proof.* Let the conditions in (1) be satisfied. Recall that $G = G'_{A,B} \cup G''_{C,D}$ with $A, B, C, D$ all nonempty. Also, $|A| + |C| = n + m - 1$ and $|B| + |D| = n + m - 2$. There is no loss of generality in assuming $n \geqslant m$ and $|A| \geqslant |C|$. We will assume $G$ contains no $P_{2n}$ and $\bar{G}$ contains no $P_{2m}$. Since all vertices in $A$ $(B)$ are adjacent in $\bar{G}$ to all vertices in $D$ $(C)$, $\min\{|A|, |D|\} \leqslant m - 1$ and $\min\{|B|, |C|\} \leqslant m - 1$. If $|A|, |C| \leqslant m - 1$, then $|A| + |C| < n + m - 1$, a contradiction. Thus $|D| \leqslant m - 1$ and $|B| \geqslant n - 1$.

If $|B| \leqslant m - 1$, then $n = m$, and $|D| = |B| = m - 1$. Since $|A| \geqslant |C|$, and $|A| \geqslant m$, there exists a $P_{2m-1}$ in $\bar{G}$ with vertices in $A$ and $D$. Therefore if there exists an edge in $\bar{G}$ with end vertices in $A$ and $B$ or $C$ and $D$, $\bar{G}$ would contain a $P_{2m}$. Thus $G'_{A,B}$ and $G''_{C,D}$ are both complete and $G$ splits.

We can assume $|B| \geqslant m$. Hence $|C| \leqslant m - 1$ and $|A| \geqslant n$. If $|B| < n$, then $|D| = m - 1$, and thus there exists a $P_{2m-1}$ in $\bar{G}$ with vertices in $A$ and $D$. Thus just as in the previous case, $G$ splits. Therefore we can assume $|B| \geqslant n$.

Since $G(P_{2n}, P_{2j}) = (n + j - 1, n + j - 1)$ for $j < m$, there exists a path in $\bar{G}'_{A,B}$ containing $2(\min\{|A|, |B|\} - n + 1)$ vertices with one end vertex in $A$ and the other in $B$. Since all edges between $A$ and $D$ ($B$ and $C$) are in $\bar{G}$, there exists a path in $\bar{G}$ containing

$$\begin{aligned} &2(\min\{|A|, |B|\} - n + 1) + 2|C| + 2|D| \\ &\quad \geqslant 2(n + m - 2 - n + 1 + 1) \geqslant 2m \end{aligned}$$

vertices. This contradiction completes the proof of (1).

The proofs of (2), (3), and (4) are very similar to the proof of (1) and thus are not included here.

LEMMA 6. *Let $P = (x_1, x_2, \ldots, x_k)$ be a maximal path in $G$. If the vertices of $P$ do not form a cycle of length $k$ and $k$ is even, then $d(x_1) + d(x_k) \leqslant k/2$.*

*Proof.* We observe that the maximality of $P$ in $G$ restricts the adjacencies of $x_1$, $x_k$ to vertices of $P$. Also, if $(x_1, x_l) \in E$, $2 \leqslant l < k$, $(x_k, x_{l-1}) \in \bar{E}$; otherwise $(x_1, x_l, x_{l+1}, \ldots, x_k, x_{l-1}, x_{l-2}, \ldots, x_1)$ is a cycle of length $k$ in $G$. Hence $d(x_1) + d(x_k) \leqslant k/2$.

LEMMA 7. *If for a fixed $n$ and $m$ with $n, m \geqslant 1$, $n + m \geqslant 3$, every $(n + m - 1, n + m - 2)$ graph $G$ is such that at least one of the following occurs,*

(i) *$G$ contains a $P_{2n}$,*

(ii) *$\bar{G}$ contains a $P_{2m}$,*

(iii) *$G$ splits,*

*then $B(P_{2n}, P_{2m}) = (n + m - 1, n + m - 1)$.*

*Proof.* If $n = 1$ or $m = 1$, it is immediate that $B(P_{2n}, P_{2m}) = (n + m - 1, n + m - 1)$. Therefore we can assume $n, m \geqslant 2$.

Let $G_1$ be an $(n + m - 1, n + m - 1)$ graph and let $x$ be a fixed vertex of $G_1$. By assumption, $G_1\backslash\{x\}$ (and hence $G_1$) contains a $P_{2n}$ or $\overline{G_1\backslash\{x\}}$ (and hence $\bar{G}_1$) contains a $P_{2m}$, unless $G_1\backslash\{x\}$ splits. Thus we can assume $G_1\backslash\{x\}$ splits, $G_1\backslash\{x\}$ contains no $P_{2n}$, and $\overline{G_1\backslash\{x\}}$ contains no $P_{2m}$. But by applying Lemma 4 we get that $G_1$ contains a $P_{2n}$ or $\bar{G}_1$ contains a $P_{2m}$ with end vertex $x$.

Let $G_2$ be an $(n + m - 1, n + m - 2)$ graph which splits into the pair $(K_{n+m-1,n-1}, K_{0,m-1})$. $G_2$ contains no $P_{2n}$ and $\bar{G}_2$ contains no $P_{2m}$. Hence $B(P_{2n}, P_{2m}) = (n + m - 1, n + m - 1)$.

THEOREM 8. *If $G$ is an $(n + m - 1, n + m - 2)$ graph with $n, m \geqslant 1$, $n + m \geqslant 3$, then at least one of the following occurs.* (i) *$G$ contains a $P_{2n}$,* (ii) *$\bar{G}$ contains a $P_{2m}$,* (iii) *$G$ splits.*

*Proof.* The proof will be by induction on the sum $n + m$. It can be checked directly that the result holds for $n = m = 2$ or if either $n = 1$ or $m = 1$. Thus let $G$ be a $(p + q - 1, p + q - 2)$ graph, $p, q \geqslant 2$, $p + q \geqslant 5$ such that the result holds for all $(n, m)$ graphs with $n + m < p + q$. By Lemma 7 and the fact that $B(P_2, P_2) = (1, 1)$, we have that $B(P_{2n}, P_{2m}) = (n + m - 1, n + m - 1)$ for $n + m < p + q$. We assume throughout the proof that $G$ contains no $P_{2p}$, and $\bar{G}$ no $P_{2q}$. Since $G = K_{p+q-1,p+q-2}$ contains a $P_{2p}$, we may assume that there exists $x, y \in G$ such that $(x, y) \in \bar{E}$. Thus choose $(x, y) \in \bar{E}$ such that $d(x) + d(y)$

is minimal. Consider $G\backslash\{x, y\}$, which is a $(p + q - 2, p + q - 3) = (p + (q - 1) - 1, p + (q - 1) - 2)$ graph. By assumption, if $G\backslash\{x, y\}$ contains no $P_{2p}$ and $\overline{G\backslash\{x, y\}}$ contains no $P_{2q-2}$, then $G\backslash\{x, y\}$ must split, i.e., $\overline{G\backslash\{x, y\}}$ splits. But by Lemma 3, if $\overline{G\backslash\{x, y\}}$ splits, since $G$ contains no $P_{2p}$, and $\bar{G}$ no $P_{2q}$, we have that $G$ must split, completing the induction. Thus we assume that $\overline{G\backslash\{x, y\}}$ contains a $P_{2q-2}$ and $\bar{G}$ contains no $P_{2q}$, i.e., that $P = (x_1, x_2, ..., x_l)$, $2q - 2 \leqslant l \leqslant 2q - 1$, is a path of maximal length in $\overline{G\backslash\{x, y\}}$.

Applying Lemmas 2 and 5(1) we have, for $l$ even, that $(x_1, x_l) \in E$ with $\bar{d}(x_1) + \bar{d}(x_l) \leqslant d(x) + d(y)$; and for $l$ odd, that $(x_1, x_{l-1}) \in E$ with $\bar{d}(x_1) + \bar{d}(x_{l-1}) < d(x) + d(y)$. Also, by the choice of $P$, $x_1$ is adjacent only to vertices of $P$ in $\bar{G}$ and not to one of $x_l, x_{l-1}$. Hence $\bar{d}(x_1) \leqslant [l/2 - 1]$. Thus in either case (whether $l$ is odd or even) there exists $u, v \in G\backslash\{x, y\}$ such that $(u, v) \in E$ and $\bar{d}(u) + \bar{d}(v) \leqslant d(x) + d(y)$ with $\bar{d}(u) \leqslant q - 2 = [l/2 - 1]$.

Next consider $G\backslash\{u, v\}$, which is again a $(p + q - 2, p + q - 3) = ((p - 1) + q - 1, (p - 1 + q - 2))$ graph. By the inductive assumption, if $G\backslash\{u, v\}$ contains no $P_{2p-2}$ and $\overline{G\backslash\{u, v\}}$ contains no $P_{2q}$, then $G\backslash\{u, v\}$ splits. But as before, if $G\backslash\{u, v\}$ splits, Lemma 3 implies $G$ splits. This completes the induction. Hence we assume that $G\backslash\{u, v\}$ contains a $P_{2p-2}$ and $G$ contains no $P_{2p}$. Let $Q = (y_1, y_2, ..., y_l)$, $2p - 2 \leqslant l \leqslant 2p - 1$ be a path of maximal length in $G\backslash\{u, v\}$. Since $\bar{d}(u) \leqslant q - 2$, $G$ a $(p + q - 1, p + q - 2)$ graph implies $d(u) \geqslant (p + q - 2) - (q - 2) = p$. But since $G$ does not contain a $P_{2p}$, $d(u) \geqslant p$ implies that there exists a $w \in G\backslash(\{u, v\} \cup Q)$ such that $(w, u) \in E$. We again apply Lemmas 2 and 5(1), this time obtaining, for $l$ even, that $(y_1, y_l) \in \bar{E}$ with $d(y_1) + d(y_l) < \bar{d}(u) + \bar{d}(v)$; and for $l$ odd, that $(y_1, y_{l-1}) \in \bar{E}$ with $d(y_1) + d(y_{l-1}) < \bar{d}(u) + \bar{d}(v)$. Thus in either case there exist $r, s \in G\backslash\{u, v\}$ such that $(r, s) \in \bar{E}$ and $d(r) + d(s) < \bar{d}(u) + \bar{d}(v) \leqslant d(x) + d(u)$, contradicting the choice of $x$ and $y$. Thus one of the assumptions made is false and the induction is complete.

Lemma 7 and Theorem 8, together with the trivial observation that $B(P_2, P_2) = (1, 1)$, give the following corollary.

COROLLARY 9. *For all* $n, m \in Z^+$, $B(P_{2n}, P_{2m}) = (n + m - 1, n + m - 1)$.

COROLLARY 10. *Each* $(n + m, n + m - 1)$ *graph* $G$ *contains a* $P_{2n+1}$ *or its complement* $\bar{G}$ *contains a* $P_{2m}$.

*Proof.* Let $G$ be an $(n + m, n + m - 1)$ graph. The result is clear when $m = 1$. If $m > 1$, then by Theorem 8, $G$ contains a $P_{2n+2}$, $\bar{G}$ contains a $P_{2m}$, or $G$ splits. Hence we may assume by Lemma 4 that $G$ splits into

$(K_{t,n}, K_{v,m-1})$, where $0 \leqslant v \leqslant \min\{n, m-1\}$. Therefore

$$t \geqslant n + m - \min\{n, m-1\} = \max\{m, n+1\}.$$

Thus $K_{t,n}$, and hence $G$, contains a $P_{2n+1}$.

COROLLARY 11. *For* $n, m \in Z^+$ *and* $n \geqslant m - 1$, $B(P_{2n+1}, P_{2m}) = (n + m, n + m - 1)$.

*Proof.* The result follows from Corollary 10 by the exhibition of $(n + m - 1, n + m - 1)$ and $(n + m, n + m - 2)$ graphs which contain no $P_{2n+1}$ and whose complements contain no $P_{2m}$. To do this, consider the $(n + m - 1, n + m - 1)$ graph $G_1$ which splits into the pair $(K_{n,n}, K_{m-1,m-1})$ and the $(n + m, n + m - 2)$ graph $G_2$ which splits into the pair $(K_{n+m,n-1}, K_{0,m-1})$.

THEOREM 12. *For all* $n \neq m$ *in* $Z^+$,

$$B(P_{2n+1}, P_{2m+1}) = (n + m, n + m - 1).$$

*Proof.* For convenience we can assume that $n > m$. We first exhibit $(n + m - 1, n + m - 1)$ and $(n + m, n + m - 2)$ graphs which contain no $P_{2n+1}$ and whose complements contain no $P_{2m+1}$. The $(n + m - 1, n + m - 1)$ graph $G_1$ which splits into the pair $(K_{n,n}, K_{m-1,m-1})$, and the $(n + m, n + m - 2)$ graph $G_2$ which splits into the pair $(K_{n+m,n-1}, K_{0,m-1})$ are such examples.

To complete the proof we need to show that each $(n + m, n + m - 1)$ graph contains a $P_{2n+1}$ or its complement contains a $P_{2m+1}$. Thus let $G$ be an $(n + m, n + m - 1)$ graph.

The proof will be by induction on $n + m$. It is clear for $n = 1$ or $m = 1$. Therefore, take $n, m > 1$, and assume that each $(i + j, i + j - 1)$ graph contains a $P_{2i+1}$ or its complement contains a $P_{2j+1}$ for $i + j < n + m$ and $i \neq j$. By Corollary 10 we may assume that $\bar{G}$ contains a $P_{2m}$ and no $P_{2m+1}$. Let $P = (x_1 x_2, ..., x_{2m})$ be a maximal path in $\bar{G}$. If $P$ can be made into a cycle, then we are done, for this implies that $\bar{G}$ contains a $P_{2m+1}$ or $\bar{G}$ separates. In the latter case Lemma 5(2) completes the result. Thus $(x_1, x_{2m}) \in E$ and by Lemma 6, $\bar{d}(x_1) + \bar{d}(x_{2m}) \leqslant m$ with $\bar{d}(x_1), \bar{d}(x_{2m}) \geqslant 1$. Choose $x, y \in G$ such that $\bar{d}(x) + \bar{d}(y)$ is minimal, with $\bar{d}(x), \bar{d}(y) \geqslant 1$, and $(x, y) \in E$. Note that $\bar{d}(x) + \bar{d}(y) \leqslant m$ and $\bar{d}(x), \bar{d}(y) \leqslant m - 1$.

Next consider $G \backslash \{x, y\}$, which is an $(n + m - 1, n + m - 2)$ graph. By Corollary 10 this $(n + m - 1, n + m - 2)$ graph contains a $P_{2n-2}$ or its complement contains a $P_{2m+1}$. Thus we assume that $G \backslash \{x, y\}$ contains a $P_l$, $2n - 2 \leqslant l \leqslant 2n$, and $G$ contains no $P_{2n+1}$. Let $G = G_{A,B}$, and $Q = (y_1, y_2, ..., y_l)$, a maximal path in $G \backslash \{x, y\}$ with $x, y_1 \in A$, $y \in B$.

First take $l = 2n - 2$. Since $\bar{d}(x)$, $\bar{d}(y) \leqslant m - 1$, $d(x)$, $d(y) \geqslant (m + n - 1) - (m - 1) = n$ with at least one of $d(x)$ or $d(y) \geqslant (n + m) - (m - 1) = n + 1$. Without loss of generality, we can take $d(y) \geqslant n + 1$. Hence there exists a $z_1 \in G\backslash(Q \cup \{x\})$ with $(z_1, y) \in E$. Then $(x, y_l) \in \bar{E}$, for otherwise $(x_1, y, x, y_l, y_{l-1}, \ldots, y_1)$ is a $P_{2n+1}$ in $G$. Hence $d(x) \geqslant n$ and $(x, y_l) \in \bar{E}$ implies there exists a $z_2 \in G\backslash\{Q \cup \{y\})$ such that $(z_2, x) \in E$. It now follows by Lemmas 2 and 5(2) that $d(y_1) + d(y_l) \leqslant \bar{d}(x) + \bar{d}(y) \leqslant m$, $d(y_1)$, $d(y_l) \geqslant 1$ and $(y_1, y_l) \in \bar{E}$.

Next take $l = 2n - 1$, so that $y_l \in A$. Just as in the previous case, $d(x)$, $d(y) \geqslant n$. Also, $(y, y_1)$, $(y, y_l) \in \bar{E}$, for otherwise $Q \cup \{y, x\}$ would be a $P_{2n+1}$. Hence there exists a $z_1 \in A\backslash(Q \cup \{x\})$ such that $(z_1, y) \in E$. Also, $(x, y_2) \in \bar{E}$, for otherwise $(z_1, y, x, y_2, \ldots, y_l)$ is a $P_{2n+1}$ in $G$. But $d(x) \geqslant n$ also implies that there exists $z_2 \in G\backslash(Q \cup \{y\})$ such that $(z_2, x) \in E$. Thus Lemmas 2 and 5(2) again apply. Hence $d(y_1) + d(y_{l-1}) < \bar{d}(x) + \bar{d}(y) \leqslant m$, $d(y_1)$, $d(y_{l-1}) \geqslant 1$, and $(y_1, y_{l-1}) \in \bar{E}$.

Finally, take $l = 2n$. This time Lemmas 2 and 5(2) apply, immediately giving $d(y_1) + d(y_l) < \bar{d}(x) + \bar{d}(y)$, $d(y_1), d(y_l) \geqslant 1$, and $(y_1, y_l) \in \bar{E}$.

Hence in all cases we can at least assume that there exist $u, v \in G\backslash\{x, y\}$, with $(u, v) \in \bar{E}$, such that $d(u) + d(v) \leqslant \bar{d}(x) + \bar{d}(y) \leqslant m < n$ and $d(u)$, $d(v) \geqslant 1$.

We finally consider the $(n + m - 1, n + m - 2)$ graph $G\backslash\{u, v\}$. By the inductive assumption (since $n > m$), $G\backslash\{u, v\}$ contains a $P_{2n+1}$ or $\overline{G\backslash\{u, v\}}$ contains a $P_{2m-1}$. Thus we assume that $P = (x_1, x_2, \ldots, x_l)$, $2m - 1 \leqslant l \leqslant 2m$, is a maximal path in $\overline{G\backslash\{u, v\}}$, and $\bar{G}$ contains no $P_{2m+1}$. Since $d(u) + d(v) < n$, by arguments similar to those just used to get $u$ and $v$, we get that there exists $w, z \in G\backslash\{u, v\}$, $(w, z) \in E$ such that $\bar{d}(w) + \bar{d}(z) < d(u) + d(v)$ and $\bar{d}(w)$, $\bar{d}(z) \geqslant 1$. But then $\bar{d}(w) + \bar{d}(z) < d(u) + d(v) \leqslant \bar{d}(x) + \bar{d}(y)$ and $\bar{d}(w), \bar{d}(z) \geqslant 1$. This contradicts the choice of $x$ and $y$, which completes the proof.

THEOREM 13. *For $n \in Z^+$, $B(P_{2n+1}, P_{2n+1}) = (2n + 1, 2n - 1)$.*

*Proof.* Consider the $(2n, 2n - 1)$ graph $G_1$, which splits into the pair $(K_{n,n}, K_{n,n-1})$, and the $(2n + 1, 2n - 2)$ graph $G_2$, which splits into the pair $(K_{2n+1,n-1}, K_{0,n-1})$. None of $G_1$, $G_2$, $\bar{G}_1$, $\bar{G}_2$ contains a $P_{n+1}$.

We complete the proof by showing that each $(2n + 1, 2n - 1)$ graph $G$ or its complements $\bar{G}$ contain a $P_{2n+1}$.

Surely each $(3, 1)$ graph or its complement contains a $P_3$, so that the result holds for $n = 1$. Hence let $n > 1$ and let $G$ be a $(2n + 1, 2n - 1)$ graph. By Corollary 10, taking $n = m$, we have that $G$ contains a $P_{2n+1}$ or $\bar{G}$ contains a $P_{2n}$. Thus we assume that $\bar{G}$ contains a $P_{2n}$ and no $P_{2n+1}$. Let $P = (x_1, x_2, \ldots, x_{2n})$ be a maximal path in $\bar{G}$. If the vertices of $P$ can be made into a cycle in $\bar{G}$, then $\bar{G}$ contains a $P_{2n+1}$ or $\bar{G}$ separates. In the latter case Lemma 5(3) shows $G$ or $\bar{G}$ contains a $P_{2n+1}$. Hence we assume that the maximal path $P$ cannot be made into a cycle. Thus $(x_1, x_{2n}) \in E$ and by Lemma 6, $\bar{d}(x_1) + \bar{d}(x_{2n}) \leqslant n$ and $\bar{d}(x_1)$, $\bar{d}(x_{2n}) \geqslant 1$. Choose $x, y \in G$ such that $\bar{d}(x) + \bar{d}(y)$ is minimal, with $\bar{d}(x)$, $\bar{d}(y) \geqslant 1$ and $(x, y) \in E$. Note that $\bar{d}(x) + \bar{d}(y) \leqslant n$, and $\bar{d}(x)$, $\bar{d}(y) \geqslant 1$ implies $\bar{d}(x), \bar{d}(y) \leqslant n - 1$.

Consider $G\backslash\{x, y\}$. Since by Theorem 12,

$$B(P_{2n-1}, P_{2n+1}) = (2n - 1, 2n - 2),$$

the proof is complete unless $G\backslash\{x, y\}$ contains a $P_{2n-1}$ and $G$ contains no $P_{2n+1}$. Thus let $Q = (y_1, y_2, ..., y_l)$, $2n - 1 \leqslant l \leqslant 2n$, be a maximal path in $G\backslash\{x, y\}$. Let $G = G_{A,B}$, with $x, y_1 \in A$ and $y \in B$.

First take $l = 2n - 1$. Since

$$\bar{d}(y) \leqslant n - 1, \qquad d(y) \geqslant (2n - 1) - (n - 1) = n.$$

Also $(y, y_1), (y, y_l) \in E$, for otherwise $G$ contains a $P_{2n+1}$. But then there exists $z_1 \in A\backslash(Q \cup \{x\})$ such that $(z_1, y) \in E$. Also $(x, y_2) \in \bar{E}$, for otherwise $(z_1, y, x, y_2, ..., y_l)$ is a $P_{2n+1}$ in $G$. But $\bar{d}(x) \leqslant n - 1$ implies $d(x) \geqslant n$. Thus (recall that $(x, y_2) \in \bar{E}$), there exists a $z_2 \in A\backslash(Q\backslash\{y\})$ such that $(z_2, x) \in E$. Applying Lemmas 2 and 5(3) we can assume that $d(y_1) + d(y_{l-1}) < \bar{d}(x) + \bar{d}(y)$, $d(y_1)$, $d(y_{l-1}) \geqslant 1$, and $(y_1, y_{l-1}) \in \bar{E}$.

Next take $l = 2n$. Then Lemmas 2 and 5(3) apply immediately, giving $d(y_1) + d(y_l) < \bar{d}(x) + \bar{d}(y)$, $d(y_1)$, $d(y_l) \geqslant 1$ and $(y_1, y_l) \in \bar{E}$.

Hence in both cases we can assume that there exist $u, v \in G\backslash\{x, y\}$ such that $d(u) + d(v) < \bar{d}(x) + \bar{d}(y)$, $d(u)$, $d(v) \geqslant 1$, and $(u, v) \in \bar{E}$.

Next we consider $G\backslash\{u, v\}$. Since by Theorem 12 $B(P_{2n+1}, P_{2n-1}) = (2n - 1, 2n - 2)$, we can, by an argument similar to the one just given, deduce that the result follows or that there exist $w, z \in G\backslash\{u, v\}$ such that $\bar{d}(w) + \bar{d}(z) < d(u) + d(v)$, $\bar{d}(w)$, $\bar{d}(z) \geqslant 1$ and $(x, z) \in E$. But then $\bar{d}(w) + \bar{d}(z) < \bar{d}(x) + \bar{d}(y)$, contradicting the choice of $x$ and $y$. This completes the proof.

THEOREM 14. *For $m - 1 > n$ in $Z^+$,*

$$B(P_{2n+1}, P_{2m}) = (n + m - 1, n + m - 1).$$

*Proof.* First observe that the $(n + m - 1, n + m - 2)$ graph $G$ splits into the pair $(K_{n+m-1,n-1}\ K_{0,m-1})$, contains no $P_{2n+1}$, and its complement $\bar{G}$ contains no $P_{2m}$.

To complete the proof we will show that for $m - 1 > n$ each $(n + m - 1, n + m - 1)$ graph contains a $P_{2n+1}$ or its complement contains a $P_{2m}$. The result is easy to check for $n = 1$, so we assume throughout that $n > 1$. Let $G$ be an $(n+m-1, n+m-1)$ graph. Since $B(P_{2n}, P_{2m}) = (n+m-1, n+m-1)$ by Corollary 9, the theorem follows unless $G$ contains a $P_{2n}$ and no $P_{2n+1}$. Assume that this is the case and let $P = (x_1, x_2, ..., x_{2n})$ be a maximal path in $G$. Now we may assume that the vertices of $P$ do not form a cycle in $G$, for otherwise $G$ contains a $P_{2n+1}$ or $G$ separates. If the latter occurs, Lemma 5(4) completes the proof. Thus by Lemma 6, $d(x_1) + d(x_{2n}) \leqslant n$. Hence one of $d(x_1)$ or $d(x_{2n})$ must be less than or equal to $n/2$. We may assume $d(x_1) \leqslant n/2$. Choose $x \in G$ such that $d(x)$ is minimal. Note that $d(x) \leqslant n/2$.

By Theorem 12, $B(P_{2n+1}, P_{2m-1}) = (n+m-1, n+m-2)$, since $n \neq m-1$. Thus since $\overline{G\backslash\{x\}}$ is an $(n+m-1, n+m-2)$ graph, $G\backslash\{x\}$ contains a $P_{2n+1}$ or $\overline{G\backslash\{x\}}$ a $P_{2m-1}$. Hence we assume $\overline{G\backslash\{x\}}$ contains a $P_{2m-1}$ and $\bar{G}$ no $P_{2m}$. Let $Q = (y_1, y_2, ..., y_{2m-1})$ be a maximal path in $\overline{G\backslash\{x\}}$.

There are two cases to consider.

*Case* I. $\bar{G} = \bar{G}_{A,B}$ with $x, y_1, y_{2m-1} \in A$.
Since $d(x) \leqslant n/2$ and $|B| = n+m-1$, we have

$$\bar{d}(x) \geqslant (n+m-1) - n/2 = n/2 + (m-1) > m-1.$$

Thus there exists a $y \in B\backslash Q$ with $(y, x) \in \bar{E}$. Applying Lemma 1 with $k = 2$, we get that $\bar{G}$ contains a $P_{2m}$ or $\bar{d}_Q(y_1) \leqslant d_Q(x)$. We thus assume $\bar{d}_Q(y_1) \leqslant d_Q(x)$. Since $Q$ is maximal in $\overline{G\backslash\{x\}}$, $\bar{d}_Q(y_1) = \bar{d}(y_1)$. Therefore $\bar{d}(y_1) \leqslant d_Q(x) \leqslant d(x)$.

*Case* II. $\bar{G} = \bar{G}_{A,B}$ with $x \in A$ and $y_1, y_{2m-1} \in B$.
Again $d(x) \leqslant n/2$ implies $\bar{d}(x) > m-1$. Also, we may assume that $(x, y_1), (x, y_{2m-1}) \in E$, for otherwise we get a $P_{2m}$ in $\bar{G}$. Therefore there exists $y \in B\backslash Q$ with $(y, x) \in \bar{E}$. Thus Lemma 1 implies that $\bar{G}$ contains a $P_{2m+1}$ or $\bar{d}(y_1) = \bar{d}_Q(y_1) < d_Q(x)$.

No matter which case occurs, we may at least assume that there exists a $u \in G\backslash\{x\}$ such that $\bar{d}(u) \leqslant d(x)$.

Next consider the $(n+m-1, n+m-2)$ graph $G\backslash\{u\}$. Since $n, m \geqslant 2$, Theorem 8 implies $G\backslash\{u\}$ contains a $P_{2n}$, $\overline{G\backslash\{u\}}$ contains a $P_{2m}$, or $G\backslash\{u\}$ splits.

First suppose that $G\backslash\{u\}$ contains no $P_{2n}$, $\overline{G\backslash\{u\}}$ contains no $P_{2m}$, and $G\backslash\{u\}$ splits. By applying Lemma 4 we get that $G\backslash\{u\}$ splits into $(K_{t,n-1}, K_{v,m-1})$, where $0 \leqslant v \leqslant n-1$ and $t = (m+n-1) - v \geqslant m$ (since $m-1 > n$). Looking at the possible adjacencies of $u$ with vertices of $G\backslash\{u\}$, we see that under the conditions just given $G$ contains a $P_{2n+1}$ or $\bar{G}$ a $P_{2m}$, a contradiction. Thus we assume that $G\backslash\{u\}$ contains a $P_{2n}$ and no $P_{2n+1}$.

Let $R = (z_1, z_2, \ldots, z_{2n})$ be a maximal path in $G \backslash \{u\}$ and $G = G_{A,B}$ with $u \in A$, $z_1 \in B$. But $\bar{d}(u) \leqslant d(x) \leqslant n/2$, so that

$$d(u) \geqslant (n + m - 1) - n/2 \geqslant (m - 1) + n/2 > n.$$

Hence there exists a $w \in B \backslash R$ such that $(w, u) \in E$. Thus Lemmas 1 and 5(4) complete the proof unless $d_R(z_1) < \bar{d}_R(u)$. Hence we assume $d_R(z_1) < \bar{d}_R(u)$. The maximality of $R$ in $G \backslash \{u\}$ implies $d(z_1) = d_R(z_1)$, so that $d(z_1) < \bar{d}_R(u) \leqslant \bar{d}(u)$.

Thus $d(z_1) < \bar{d}(u) \leqslant d(x)$, a contradiction to the choice of $x$. This completes the proof of the theorem.

Another proof of some of these results, appear in an article entitled "Ramsey type problems for paths by A. Dyárfás and J. Lehel in *Periodica Mathematica Hungarica* **3** (1973), 299–304.

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